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Two-sided purity on regular semigroups

by

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1. A subsemigroup B of a semigroup S is called a *bi-ideal* of S if

$$BSB \subseteqq B .$$

The author introduced in [2] the notion of two-sided purity of bi-ideals of a semigroup; A bi-ideal B of a semigroup S is called *two-sided pure* or simply *T-pure* if

$$B \cap xSy = xBy$$

holds for all elements x and y of S. A semigroup S is called *T^*-pure* if every bi-ideal of it is T-pure. In this paper we shall give some properties of T-pure ideals of a regular semigroup and prove that a semigroup is regular and T^*-pure if and only if it is a semilattice of groups.

2. Let S be a semigroup. Following the notation and terminology of A. H. Clifford and G. B. Preston [1] we shall say that S is a *semilattice of groups* if it is the set-theoretical union of a family $\{S_\alpha : \alpha \in M\}$ of mutually disjoint subgroups S_α such that, for every $\alpha, \beta \in M$, the products $S_\alpha S_\beta$ and $S_\beta S_\alpha$ are both contained in the same subgroup S_γ $(\gamma \in M)$. A semigroup S is called *regular* if, for every element a of S, there exists an element x in S such that $a = axa$.

Now we list some results which will be needed in the following discussions.

LEMMA 1 ([4] Theorem 4). *A semigroup S is a semilattice of groups if and only if the condition*

$$B \cap X = BXB$$

holds for all bi-ideals B and all one-sided ideals X of S.

LEMMA 2 ([4] Theorem 12). *Let S be a semigroup which is a semilattice of groups. Then every bi-ideal of S is a two-sided ideal of S.*

LEMMA 3 ([2] Lemma 4). *Let B be any T-pure bi-ideal of a semigroup S. Then*

$$B \cap XSY = XBY$$

holds for all bi-ideals X and Y of S.

LEMMA 4 ([6] Theorem). *A semigroup S is regular if and only if the condition*

$$B=BSB$$

holds for all bi-ideals B of S.

Our regularity criterion for semigroups reads as follows.

THEOREM 5. *A semigroup S is regular if and only if the condition*

$$B\cap I=BIB$$

holds for all bi-ideals B and all ideals I of S.

Proof. Suppose that S is regular. Let B and I be respectively any bi-ideal and ideal of S, and let a be any element of $B\cap I$. Since regular, there exists an element x in S such that $a=axa$. Then we S is have

$$a=axa=axaxa\in BSISB\subseteqq BIB$$

and so we have

$$B\cap I\subseteqq BIB .$$

On the other hand, we have

$$BIB\subseteqq BSB\subseteqq B \text{ and } BIB\subseteqq I .$$

Thus we have

$$BIB\subseteqq B\cap I .$$

Therefore we obtain that

$$B\cap I=BIB$$

for all bi-ideals B and all ideals I of S.

Conversely, suppose that the condition

$$B\cap I=BIB$$

holds for all bi-ideals B and all ideals I of S. In case of $I=S$, the above equality implies that

$$B=B\cap S=BSB .$$

Then it follows from Lemma 4 that S is regular. This completes the proof of the theorem.

3. We note that any ideal of a semigroup is a bi-ideal. In this section we shall give some properties of T-pure ideals.

THEOREM 6. *Any ideal of a regular semigroup is T-pure.*

Proof. Let I be any ideal of a regular semigroup S and x and y any elements of S. Since xSy is a bi-ideal of S, by Theorem 5 we have

$$I \cap xSy = (xSy)I(xSy) = x(SyIxS)y \subseteqq xIy \ .$$

Since I is an ideal of S, the converse inclusion always holds. Thus we obtain that

$$I \cap xSy = xIy$$

holds for all $x, y \in S$, and that I is T-pure. This completes the proof of the theorem.

We denote by $L[x]$, $R[x]$ and $B[x]$ the principal left ideal, right ideal and bi-ideal of a semigroup S generated by x in S.

The next theorem shows that the converse of Lemma 3 holds in the case when the bi-ideal B is a two-sided ideal. The equivalence of (2) and (3) in the next theorem is due to the author ([2] Lemma 3).

THEOREM 7. *For any ideal A of a semigroup S the following conditions are equivalent.*

(1) *A is T-pure.*
(2) *$A \cap XSY = XAY$ for all bi-ideals X and Y of S.*
(3) *$A \cap B[x]SB[y] = B[x]AB[y]$ for all $x, y \in S$.*
(4) *$A \cap R[x]SL[y] = R[x]AL[y]$ for all $x, y \in S$.*

Proof. It is clear that (2) implies (4). Thus by Lemma 3 it suffices to prove that (3) implies (1) and that (4) implies (1). Assume that (4) holds. Let x and y be any elements of S and $a = xsy$ ($a \in A$, $s \in S$) any element of $A \cap xSy$. Then we have

$$\begin{aligned} a = xsy &\in A \cap R[x]SL[y] \\ &= R[x]AL[y] = (x \cup xS)A(y \cup Sy) \\ &= xAy \cup xA(Sy) \cup (xS)Ay \cup (xS)A(Sy) \\ &= xAy \cup x(AS)y \cup x(SA)y \cup x(SAS)y \\ &\subseteqq xAy \end{aligned}$$

and so we have

$$A \cap xSy \subseteqq xAy \ .$$

Since the converse inclusion always holds, we obtain that

$$A \cap xSy = xAy$$

holds for all $x, y \in S$, and that (4) implies (1).

We note that

$$B[x] = x \cup x^2 \cup xSx$$

holds for every elements x of S ([1] p. 84). Then similarly we can prove that (3) implies (1). This completes the proof of the theorem.

Remark 1. It can be found another conditions which are equivalent to any one of the conditions (1)~(4) of Theorem 7.

For example,

(5) $A \cap B[x]Sy = xAL[y]$ *for all* $x, y \in S$.

THEOREM 8. *For any ideal* A *of a regular semigroup* S *the following conditions are equivalent*

(1) A *is T-pure.*

(2) $A \cap eSf = eAf$ *for all idempotent elements* e *and* f *of* S.

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let x and y be any elements of S. Then, since S is regular, there exist elements x' and y' in S such that

$$x = xx'x \text{ and } y = yy'y$$

and both xx' and $y'y$ are idempotent. Then we have

$$\begin{aligned} A \cap xSy &= A \cap (xx'x)S(yy'y) \\ &= A \cap (xx')(xSy)(y'y) \subseteqq A \cap (xx')S(y'y) \\ &= (xx')A(y'y) = x(x'Ay')y \subseteqq xAy \,. \end{aligned}$$

Since the inclusion

$$A \cap xSy \supseteqq xAy$$

always holds, we obtain that

$$A \cap xSy = xAy$$

holds for all $x, y \in S$, and that (2) implies (1). This completes the proof of the theorem.

Remark 2. It can be found another conditions which are equivalent to any one of the conditions (1)~(2) of Theorem 8.

For example,

(3) $A \cap B[e]Sy = eAL[y]$ *for all idempotent elements* e *and all elements* y *of* S.

4. In this section we shall give a characterization of semigroups, which are semilattices of groups. For another characterizations of such semigroups, see S. Lajos [3] and [4].

THEOREM 9. *For a semigroup* S *the following conditions are equivalent.*

(1) S *is a semilattice of groups.*

(2) S *is regular and* T^**-pure.*

Proof. Suppose that S is a semilattice of groups. Let B be any bi-ideal of S. Then, since S itself is a one-sided ideal of S, by Lemma 1 we have

$$BSB=B\cap S=B\ .$$

Then it follows from Lemma 4 that S is regular (cf. [3] Theorem 3 (W)). On the other hand, by Lemma 2, B is an ideal of S. Then it follows from Theorem 6 that B is T-pure. Since B is any bi-ideal of S, we obtain that S is T^*-pure, and that (1) implies (2).

Conversely, suppose that (2) holds. Let B and X be respectively any bi-ideal and any one-sided ideal of S. Then, since S is regular, it follows from Lemma 4 that

$$B=BSB\ .$$

Since the one-sided ideal X is a T-pure bi-ideal of S, by Lemma 3 we have

$$BXB=X\cap BSB=X\cap B\ .$$

Then it follows from Lemma 1 that S is a semilattice of groups.

This completes the proof of the theorem.

Following [1] we shall say that a semigroup S is a *left group* if it is left simple and right cancellative. Dually, S is called a *right group* if it is right simple and left cancellative. Recently S. Lajos gave the following.

LEMMA 10 ([5] Theorem 1). *For a semigroup S the following conditions are equivalent.*

(1) *S is a semilattice of left groups.*

(2) *$B\cap L=BLB$ for all bi-ideals B and all left ideals L of S.*

For T^*-pure semigroups we have the following.

THEOREM 11. *The following assertions concerning a T^*-pure semigroup S are mutually equivalent.*

(1) *S is regular.*

(2) *S is a semilattice of left groups.*

(3) *S is a semilattice of right groups.*

(4) *S is a semilattice of groups.*

Proof. It follows from Theorem 9 that (1) and (4) are equivalent. It is clear that (4) implies (2) and (3). It follows from Lemmas 4 and 10 that (2) implies (1). Similarly it can be proved that (3) implies (1). This completes the proof.

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